Probability Theory

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Chapter 07: Properties of Expectation

Expectation of a Function of Random Variables

Proposition

Suppose that X and Y are RVs and g is a function of the two variables.

If X and Y have a joint pmf $p(x, y)$,

$$E[g(X,Y)] = \sum_Y \sum_X g(x,y)p(x,y)$$

If X and Y have a joint pdf $f(x, y)$,

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\ \mathrm{d}x\,\mathrm{d}y$$

Example

An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expected distance between the ambulance and the point of the accident.

Solution

$$f(X,Y) = 1/L^2, \qquad 0 < x < L, \quad 0 < y < L$$

$$E[|X-Y|] = \frac{1}{L^2}\int_0^L\int_0^L |x-y|\ \mathrm{d}x\,\mathrm{d}y$$

$$\int_0^L |x-y|\ \mathrm{d}x = \int_0^y (y-x)\ \mathrm{d}x + \int_y^L (x-y)\ \mathrm{d}x = \tfrac{1}{2}L^2 - Ly + y^2$$

$$E[|X-Y|] = \frac{1}{L^2}\int_0^L \left(\tfrac{1}{2}L^2 - Ly + y^2\right)\ \mathrm{d}y = \frac{L}{3}$$

Expectation of Sums of Random Variables

In the continuous case

$$\begin{aligned} E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\ \mathrm{d}y\,\mathrm{d}x + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} xf_X(x)\ \mathrm{d}x + \int_{-\infty}^{\infty} yf_Y(y)\ \mathrm{d}y \\ &= E[X] + E[Y] \end{aligned}$$

The same result holds in the discrete case.

In general,

We may show by a simple induction proof that if $E[X_i]$ is finite for all $i = 1, 2, \cdots, n$, then

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

Example

The sample mean

Let $X_1, X_2, \cdots, X_n$ be i.i.d. RVs having distribution function F and expected value μ. Such a sequence of RVs is said to constitute a sample from the distribution F. Compute the expected value of the sample mean, $E[\overline{X}]$, where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Solution

$$\begin{aligned} E[\overline{X}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu \\ &= \mu \end{aligned}$$

Expectation of Products of Independent RVs

Proposition

If X and Y are independent, then, for any functions h and g,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof

Suppose that X and Y are jointly continuous with joint density $f(x, y)$. Then

$$\begin{aligned} E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} g(x)f_X(x)\ \mathrm{d}x \int_{-\infty}^{\infty} h(y)f_Y(y)\mathrm{d}y \\ &= E[g(X)]E[h(Y)] \end{aligned}$$

The proof in the discrete case is similar.

Covariance

Definition

The covariance between X and Y, denoted by $\mathrm{Cov}(X,Y)$, is defined by

$$\mathrm{Cov}(X,Y) = E\left[(X - E[X])\,(Y - E[Y])\right]$$

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E\left[(X - E[X])\,(Y - E[Y])\right]\\ &= E\left[XY - E[X]Y - XE[Y] + E[X]E[Y]\right]\\ &= E[XY] - E\left[E[X]Y\right] - E\left[XE[Y]\right] + E\left[E[X]E[Y]\right]\\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y]\\ &= E[XY] - E[X]E[Y]\end{aligned}$$

Covariance and Independence

If X and Y are independent

$$\mathrm{Cov}(X,Y) = 0$$

The converse is not true!

Example

Let X be a RV such that

$$P\{X=0\} = P\{X=1\} = P\{X=-1\} = \frac{1}{3}$$

and defining

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

Now, $XY = 0$, so $E[XY] = 0$. Also, $E[X] = 0$. Thus,

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0$$

However, X and Y are clearly not independent.

Properties of Covariance

Proposition

1. $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
2. $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$
3. $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$
4. $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$

Proof

3. $$\begin{aligned}\mathrm{Cov}(aX,Y) &= E[aXY] - E[aX]E[Y]\\ &= aE[XY] - aE[X]E[Y]\\ &= a\left(E[XY] - E[X]E[Y]\right)\\ &= a\,\mathrm{Cov}(X,Y)\end{aligned}$$

Properties of Covariance (cont'd)

Proof

4. Let $\mu_i = E[X_i]$ and $\nu_j = E[Y_j]$. Then

$$\begin{aligned}\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) &= E\left[\left(\sum_{i=1}^{n} X_i - E\left[\sum_{i=1}^{n} X_i\right]\right)\left(\sum_{j=1}^{m} Y_j - E\left[\sum_{j=1}^{m} Y_j\right]\right)\right]\\ &= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} E[X_i]\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} E[Y_j]\right)\right]\\ &= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} \nu_j\right)\right]\\ &= E\left[\sum_{i=1}^{n} (X_i - \mu_i) \sum_{j=1}^{m} (Y_j - \nu_j)\right]\end{aligned}$$

Properties of Covariance (cont'd)

Proof (cont'd)

4. $\cdots$. Therefore,

$$\begin{aligned}\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) &= E\left[\sum_{i=1}^{n} (X_i - \mu_i) \sum_{j=1}^{m} (Y_j - \nu_j)\right]\\ &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m} (X_i - \mu_i)(Y_j - \nu_j)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} E\left[(X_i - \mu_i)(Y_j - \nu_j)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}\left(X_i, Y_j\right)\end{aligned}$$

Thus, the covariance operation is additive.

Variance of Sums of Random Variables

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i<j}\sum \mathrm{Cov}\left(X_i, Y_j\right)$$

Proof

From parts 2 and 4 of the last proposition, upon taking $Y_j = X_j$, $j = 1, 2, \cdots, n$,

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)\\ &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i\neq j}\sum \mathrm{Cov}\left(X_i, X_j\right)\end{aligned}$$

Each pair of indices i, j, $i \neq j$, appears twice in the double summation.

Variance of Sums of Independent Random Variables

If $X_1, X_2, \cdots, X_n$ are pairwise independent, in that X_i and X_j are independent for $i \neq j$, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

Example: Variance of sample mean

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables having expected value μ and variance σ^2. Find the variance of the sample mean, $\mathrm{Var}(\overline{X})$.

Solution

$$\mathrm{Var}(\overline{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \qquad = \frac{\sigma^2}{n}$$

Example: The Sample Variance

Let $X_1, X_2, \cdots, X_n$ be i.i.d. RVs having expected value μ and variance σ^2. The quantities $X_i - \overline{X}$, $i = 1, 2, \cdots, n$, are called *deviations*, as they equal the differences between the individual data and the sample mean, $\overline{X}$. The random variable

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

is called the *sample variance*. Find the $E[S^2]$.

Solution

$$(n-1)S^2 = \sum_{i=1}^{n}\left(X_i - \mu + \mu - \overline{X}\right)^2$$

$$(n-1)S^2 = \sum_{i=1}^{n}\left((X_i - \mu) - (\overline{X} - \mu)\right)^2$$

$$= \sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)\sum_{i=1}^{n}(X_i - \mu)$$

Example: The Sample Variance (cont'd)

Solution (cont'd)

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)\sum_{i=1}^{n}(X_i - \mu)$$

$$= \sum_{i=1}^{n}(X_i - \mu)^2 + n(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)n(\overline{X} - \mu)$$

$$= \sum_{i=1}^{n}(X_i - \mu)^2 - n(\overline{X} - \mu)^2$$

Taking expectations of the preceding yields

$$(n-1)E[S^2] = \sum_{i=1}^{n} E\left[(X_i - \mu)^2\right] - nE\left[(\overline{X} - \mu)^2\right]$$

$$= n\sigma^2 - n\mathrm{Var}(\overline{X}) \qquad (\text{since } E[\overline{X}] = \mu)$$

$$= n\sigma^2 - \sigma^2 \qquad = (n-1)\sigma^2$$

$$E[S^2] = \sigma^2$$

Correlation

Definition

The correlation of X and Y, denoted by $\rho(X,Y)$, is defined, as long as $\mathrm{Var}(X)\mathrm{Var}(Y)$ is positive, by

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

Properties of Correlation

1. $-1 \leq \rho(X,Y) \leq 1$
2. $\rho(X,Y) = \pm 1 \iff Y = a \pm bX$

Interpretation

The correlation coefficient is a measure of the degree of linearity between X and Y. A value of $\rho(X,Y)$ near ± 1 indicates a high degree of linearity, whereas a value near 0 indicates that such linearity is absent. A positive value of $\rho(X,Y)$ indicates that Y tends to increase when X does, whereas a negative value indicates that Y tends to decrease when X increases. If $\rho(X,Y) = 0$, then X and Y are said to be *uncorrelated*.

Example

Let $X_1, X_2, \cdots, X_n$ be i.i.d. RVs having variance σ^2. Show that

$$\mathrm{Cov}(X_i - \overline{X}, \overline{X}) = 0$$

Solution

$$\mathrm{Cov}(X_i - \overline{X}, \overline{X}) = \mathrm{Cov}(X_i, \overline{X}) - \mathrm{Cov}(\overline{X}, \overline{X})$$

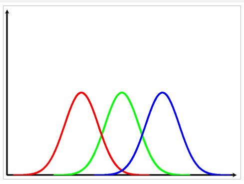
$$= \mathrm{Cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) - \mathrm{Var}(\overline{X})$$

$$= \frac{1}{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i, X_j) - \mathrm{Var}(\overline{X})$$

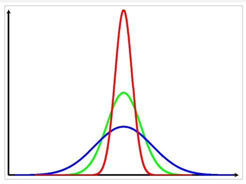
$$= \frac{1}{n}\sum_{j \neq i}\mathrm{Cov}(X_i, X_j) + \frac{1}{n}\mathrm{Var}(X_i) - \mathrm{Var}(\overline{X})$$

$$= 0 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \qquad = 0$$

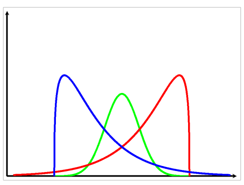
Significance of Moments



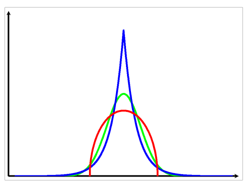
First Moment



Second Central Moment



Third Standardized Moment



Fourth Standardized Moment

Moments of the Number of Events that Occur

Let $A_1, A_2, \cdots, A_n$ be events. An indicator variable I_i is defined for event A_i such that

$$I_i = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

Let X be the number of these events that occur. Therefore,

$$X = \sum_{i=1}^{n} I_i$$

$$E[X] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} P(A_i)$$

Moments of the Number of Events that Occur

Suppose we are interested in the number of *pairs* of events that occur.

$$\binom{X}{2} = \sum_{i<j} I_i I_j$$

Taking expectations yields

$$E\left[\binom{X}{2}\right] = E\left[\sum_{i<j} I_i I_j\right] = \sum_{i<j} E[I_i I_j] = \sum_{i<j} P(A_i A_j)$$

or

$$E\left[\frac{X(X-1)}{2}\right] = \sum_{i<j} P(A_i A_j)$$

$$E[X^2] - E[X] = 2\sum_{i<j} P(A_i A_j)$$

which yields $E[X^2]$.

Moments of the Number of Events that Occur

By considering the number of distinct subsets of k events that all occur, we see that

$$\binom{X}{k} = \sum_{i_1<i_2<\cdots<i_k} I_{i_1} I_{i_2} \cdots I_{i_k}$$

Taking expectations yields

$$E\left[\binom{X}{k}\right] = E\left[\sum_{i_1<i_2<\cdots<i_k} I_{i_1} I_{i_2} \cdots I_{i_k}\right] = \sum_{i_1<i_2<\cdots<i_k} E\left[I_{i_1} I_{i_2} \cdots I_{i_k}\right] = \sum_{i_1<i_2<\cdots<i_k} P(A_{i_1} A_{i_2} \cdots A_{i_k})$$

which yields $E[X^k]$ in terms of previous moments.

Example

Moments of binomial random variables

Consider n independent trials, with each trial being a success with probability p. Let A_i be the event that trial i is a success.

$$E\left[\binom{X}{k}\right] = \sum_{i_1<i_2<\cdots<i_k} P(A_{i_1} A_{i_2} \cdots A_{i_k}) = \sum_{i_1<i_2<\cdots<i_k} p^k = \binom{n}{k} p^k$$

or, equivalently,

$$E\left[X(X-1)\cdots(X-k+1)\right] = n(n-1)\cdots(n-k+1)p^k$$

$$E[X] = np$$

$$E[X(X-1)] = n(n-1)p^2$$

$$E[X^2 - X] = n(n-1)p^2$$

$$E[X^2] = n(n-1)p^2 + np$$

$$E[X(X-1)(X-2)] = n(n-1)(n-2)p^3$$

$$E[X^3 - 3X^2 + 2X] = n(n-1)(n-2)p^3$$

$$E[X^3] = n(n-1)(n-2)p^3 + 3E[X^2] - 2E[X] = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

Moment Generating Functions

Definition

The moment generating function $M(t)$ of the RV X is defined by

$$M(t) = E[e^{tX}], \qquad t \in \mathbb{R}$$

Generating the moments

When $M(t)$ *exists*, all of the moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.

$$M'(t) = \begin{cases} \frac{d}{dt} \sum_x e^{tx} p(x) = \sum_x x e^{tx} p(x) & \text{disc.RV} \\ \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx & \text{cont.RV} \end{cases} = E[Xe^{tX}]$$

$$M''(t) = E[X^2 e^{tX}]$$

$$M^{(n)}(t) = E[X^n e^{tX}]$$

$$E[X^n] = M^{(n)}(0), \qquad n \geq 1$$

Example: Binomial Distribution

$$X \sim \text{Binomial}(n, p)$$

$$M(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n$$

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

$$E[X^2] = M''(0) = n(n-1)(pe^0 + 1 - p)^{n-2}(pe^0)^2 + n(pe^0 + 1 - p)^{n-1} pe^0 = n(n-1)p^2 + np$$

$$M'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$E[X] = M'(0) = n(pe^0 + 1 - p)^{n-1} pe^0 = np$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p)$$

Example: Exponential Distribution

$$X \sim \text{Exponential}(\lambda)$$

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx \\ &= \lambda\int_0^\infty e^{-(\lambda-t)x}\,dx \\ &= \frac{\lambda}{\lambda-t}, \quad \text{for } \lambda > t \end{aligned}$$

$$\begin{aligned} M'(t) &= \frac{\lambda}{(\lambda-t)^2} \\ E[X] &= M'(0) \\ &= \frac{\lambda}{(\lambda-0)^2} = \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} M''(t) &= \frac{2\lambda}{(\lambda-t)^3} \\ E[X^2] &= M''(0) \\ &= \frac{2\lambda}{(\lambda-0)^3} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

Example: Normal Distribution

$$Z \sim \text{Normal}(0,1)$$

$$\begin{aligned} M_Z(t) &= E[e^{tZ}] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2-2tx}{2}\right\}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\frac{(x-t)^2}{2}+\frac{t^2}{2}\right\}dx \\ &= e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,dx \\ &= e^{t^2/2} \end{aligned}$$

$$X \sim \text{Normal}(\mu,\sigma^2)$$

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= E[e^{t(\mu+\sigma Z)}] \\ &= E[e^{t\mu}e^{t\sigma Z}] \\ &= e^{t\mu}E[e^{t\sigma Z}] \\ &= e^{t\mu}M_Z(t\sigma) \\ &= e^{t\mu}e^{(t\sigma)^2/2} \\ &= \exp\left\{\frac{\sigma^2t^2}{2}+\mu t\right\} \end{aligned}$$

Properties of Moment Generating Functions

MGF of Sums of Indep. RVs

The moment generating function of the sum of independent RVs equals the product of the individual moment generating functions.

Proof

Let X and Y be indep. RVs having MGF's $M_X(t)$ and $M_Y(t)$, respectively.

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

Uniqueness Property of MGFs

If $M_X(t)$ exists and is finite in some region about $t=0$, then the distribution of X is uniquely determined.

Example

$$M_X(t) = \left(\frac{1}{2}\right)^{10}\left(e^t+1\right)^{10} \quad \Longleftrightarrow \quad X \sim \text{Binomial}(10,\tfrac{1}{2})$$

Example

If X and Y are indep. normal RVs with respective parameters (μ_1,σ_1^2) and (μ_2,σ_2^2). Find the distribution of $X+Y$.

Solution

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= \exp\left\{\frac{\sigma_1^2t^2}{2}+\mu_1t\right\}\exp\left\{\frac{\sigma_2^2t^2}{2}+\mu_2t\right\} \\ &= \exp\left\{\frac{(\sigma_1^2+\sigma_2^2)t^2}{2}+(\mu_1+\mu_2)t\right\} \end{aligned}$$

$$X+Y \sim \text{Normal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

Example

Compute the MGF of a chi-squared RV, χ_n^2, with n degrees of freedom.

Solution

$$\chi_n^2 = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

where $Z_1, Z_2, \cdots, Z_n$ are indep. standard normal RVs.

$$\begin{aligned} M_{Z^2}(t) &= E[e^{tZ^2}] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx^2}e^{-x^2/2}\,dx \\ &= \sigma \times \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx \\ &= \sigma \end{aligned}$$

where $\sigma^2 = 1/(1-2t)$

$$\begin{aligned} M_{\chi_n^2}(t) &= (M_{Z^2}(t))^n \\ &= \sigma^n \\ &= \left(\frac{1}{\sqrt{1-2t}}\right)^n \end{aligned}$$